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On the origin of wave patterns in fluidized beds

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In this report, a model designed for the description of the flow of two miscible phases in a fluidized bed is discussed. Apart from basic problems of modelling accurately such multi-phase flows, little analytical progress had been achieved in the investigation of a certain standard model based on the theory of interacting continua. It turns out, however, that the model under consideration can be investigated with the help of bifurcation theory. In particular, the methods of the theory of bifurcation with symmetry can be applied owing to the symmetries of the system.

In general, a stationary homogeneous state exists in fluidized beds which can become unstable when the physical parameters of the system are varied. Then pattern formation takes place, e.g. in the form of one- and/or two-dimensional waves, bubbles, or convection patterns; also turbulent behaviour has been observed.

In order to understand the occurrence of wave patterns and other phenomena as an inherent feature of the system, a finite, but periodically continued two-dimensional bed is investigated. While this suppresses certain boundary effects, it gives us thorough insight into the principal behaviour of this complicated system.

In particular, it allows us not only to perform easily a linear stability analysis of the basic state of uniform fluidization, but also to conclude that bifurcation of travelling waves occurs when this state becomes unstable. Well-known patterns like vertical and oblique travelling waves (OTW) of the form $u(x,y,t) = \tilde{u}(x-\omega t \pm ky)$, $k \geqslant 0$, are discovered. Owing to symmetry, the existence of standing travelling waves (STW) of the form $u(x,y,t) = \tilde{u}(x-\omega t, y)$ is also expected, but regrettably no mathematically rigorous proof of this last conjecture is presently available.

Bubble formation can also be approached via the instability of a vertical travelling plane wave train to transverse perturbations. Then a secondary stationary bifurcation to another kind of standing travelling waves takes place. This scenario is also in agreement with experimental observations. In addition, the occurrence of bifurcations of higher order, which lead to more and more complex wave patterns and are to be found on the route to turbulence, can be deduced.

1. Introduction

A fluidized bed consists of a collocation of solid particles which is subject to a vertical, upward flow of fluid or gas. On increasing the fluid flow rate slowly, a state is reached where the particles are free to move owing to the balance of the downward gravitational force with the upward drag exerted by the fluid. This state, where the behaviour of the system is similar to that of fluids, is called 'fluidization'. Increasing the flow rate above that of minimum fluidization will cause the bed to expand uniformly until a critical flow rate is reached, at which inhomogeneities like 'bubbles' or 'slugs' develop; also plane waves propagating upwards through the bed and thereby developing a transversal structure have been observed. In changing the parameters, especially the entrance velocity of the fluid and with it the flow rate, both bubble and turbulent flows can be generated (with diverse transitions or successions, respectively) (Didwania & Homsy 1981*a*).

Although fluidized beds have been investigated and used for a long time, many aspects of the motions of the particular phases are still unclear, and the development of the corresponding theory has made only slow progress. For the past one or two decades two-phase flows have been described as interpenetrating and interacting continua (Jackson 1971; Garg & Pritchett 1975; Drew 1983). The treatment of such models, however, did not go much beyond very simplifying considerations; in most papers only the equations linearized at the homogeneous basic flow were investigated for different geometries and boundary conditions (Medlin, Wong & Jackson 1974; Spiegel & Childress 1975; Didwania & Homsy 1981*b*). To our knowledge, Spiegel & Childress (1975) were the first to suggest that bifurcation methods might be applied successfully to this problem. In another approach referring to Benjamin–Feir instabilities, Didwania & Homsy (1982) computed the possible instability of a plane wave train to transverse perturbations as a resonance effect of the first harmonic of the one-dimensional wave with 'sidebands' (small transverse wavenumbers).

Further nonlinear treatments of the problem have been carried out almost exclusively in one space dimension. Kluwick (1983) used singular perturbation theory to derive simplified equations, which yield either shock or travelling wave solutions depending on whether viscosity is taken into account or not. Similar approaches were used recently by Ganser & Drew (1987, 1990), and Kurdyumov & Sergeev (1987) and Sergeev (1990).

After a summarizing analysis of the diverse terms occurring in the linearized theory and their meaning within a weakly nonlinear theory (Liu 1982), Liu (1983) tried to prove the existence of a non-trivial state different from the basic state, and that the new state is stable when the basic state is unstable. Using a slightly simplified model, Needham & Merkin (1983, 1984*b*, 1986) demonstrated that a periodic travelling wave bifurcates from the basic state, if the latter becomes unstable. Moreover, their numerical simulations showed waves, whose structures are strongly reminiscent of slug flows. Two-dimensional though linear calculations of the same authors indicate a bubbly flow (Needham & Merkin 1984*a*). Pritchett, Blake & Garg (1978) studied numerically slightly different equations in two space dimensions and observed the development of bubbles at the orifices on the ground where the fluidizing gas or fluid enters the bed.

In the following sections an extended version of the model of Garg & Pritchett (1975) will be considered from a bifurcation point of view. Thus, the aforementioned approximate investigations can be put onto a firm basis. Furthermore, these results can be extended in various directions. Assuming periodicity in space, we will show that bifurcations to time-periodic solutions can occur when the basic state of homogeneous fluidization becomes unstable. These are either oblique travelling or 'standing' travelling waves of certain symmetries. The latter in particular are very reminiscent of the familiar but yet unexplained bubbles, which occur frequently in fluidized beds (figure 1).

We admit that there is no general consensus concerning the proper modelling of fluidized beds as is expressed once more by the rather different approach of Batchelor (1988). The outcome of our analysis, however, shows that the model investigated does make sense and may yield an explanation of the various wave patterns observed

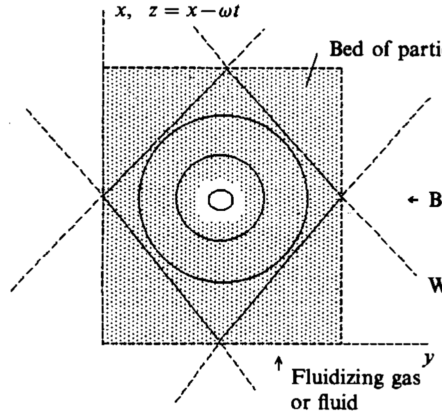

FIGURE 1. Oblique travelling and standing travelling waves and their relation to bubbling in fluidized beds; coordinate system.

in two-phase flows. Moreover, Hernandez & Jimenez (1991) recently presented numerical calculations based on a similar model. Here a bubble of a quite realistic shape was obtained as a secondary instability. This agrees with our predictions (Göz 1990*a*).

The model under consideration is described briefly in §2, followed by a compact presentation of the results of the linear stability analysis in §3. In §4 a Galilei-transformed coordinate system moving with the velocity of the anticipated travelling wave solutions will be described. The analysis of these new equations linearized at the basic state indicates the bifurcation of spatio-temporal oscillating states. Whereas the bifurcation of oblique travelling waves can be proved in two different ways, the existence of the standing travelling waves remains unclear due to some serious mathematical problems connected with models of this kind. Together with some resolution possibilities this problem will be discussed at the end of §4. In §5 there is a survey of bifurcations of higher order which lead to more and more complex flow patterns and lie on the route to turbulence. Finally, in our conclusions we summarize our results obtained so far and outline further unsolved problems.

2. Description of the model

First we introduce the model equations and define the basic solution. Then this solution is used to write the equations in dimensionless form.

2.1. *The basic equations*

We study the pure hydrodynamic problem (i.e. without any chemical reactions, etc.) of the interaction of two interpenetrating media in a fluidized bed, where the first medium is a gas or fluid and the second consists of small solid particles. It is quite common to consider the particle phase as a continuum also (Garg & Pritchett 1975), e.g. after carrying out an appropriate averaging process (Jackson 1971), although the interactions between the phases need to be determined either empirically or by means of other modelling considerations and are given at this stage only formally. For further simplification it is assumed that the two phases behave like incompressible Newtonian fluids, so that density changes are only due to changes in the volume fractions of the two phases. Thus, for each single phase the compressible

Navier–Stokes equations (with interaction terms) are assumed to hold, whereas the system as a whole behaves like an incompressible fluid in the sense that the mean flow is divergence-free.

Let the volume fraction of the fluid (or gas) be denoted by ϕ; then the relative-densities are given by

$$\rho_1 = \rho_s(1-\phi), \quad \rho_2 = \rho_f \phi, \quad \phi \in (0,1) \tag{2.1}$$

with ρ_s, ρ_f the constant specific densities of the solid and fluid, respectively. Effective fluid and particle pressures are introduced (Garg & Pritchett 1975):

$$p_f = \phi p_e, \quad p_s = (1-\phi)\, p_{se}, \tag{2.2a}$$

and the following relation between these pressures is proposed:

$$p_{se} = p_e + g(\phi). \tag{2.2b}$$

According to Garg & Pritchett (1975), the function $g(\phi)$ represents the normal component of the particle–particle interaction and has to be determined empirically. For incompressible particles it is supposed to decrease monotonically with ϕ and tend to the following limits:

$$g'(\phi) < 0 \quad \text{for all} \quad \phi \in (0,1), \qquad \left.\begin{array}{ll} \phi \to 0: & g(\phi) \to \infty, \\ \phi \to 1: & g(\phi) \to 0. \end{array}\right\} \tag{2.2c}$$

Using this ansatz, the following conservation laws are obtained for mass and momentum:

$$\partial_t(1-\phi) + \nabla\cdot[(1-\phi)\,\boldsymbol{v}] = 0 \quad \text{(particles)}, \tag{2.3a}$$

$$\partial_t \phi + \nabla\cdot(\phi \boldsymbol{u}) = 0 \quad \text{(fluid)}, \tag{2.3b}$$

$$\rho_s(1-\phi)\,(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = \nabla\cdot\boldsymbol{\sigma}_s + (1-\phi)\,\rho_s \boldsymbol{g} + \boldsymbol{F} \quad \text{(particles)}, \tag{2.3c}$$

$$\rho_f \phi(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = \nabla\cdot\boldsymbol{\sigma}_f + \phi\rho_f \boldsymbol{g} - \boldsymbol{F} \quad \text{(fluid)}, \tag{2.3d}$$

where ϕ is the voidage, $\boldsymbol{v}$ and $\boldsymbol{u}$ are the velocities of the solid and fluid phases, respectively, $\boldsymbol{g}$ is the gravity force, $\boldsymbol{\sigma}_{f,s}$ are the strain-stress tensors of the fluid/solid phases, and $\boldsymbol{F}$ represents interaction forces. The latter are assumed to be of the form

$$\boldsymbol{F} = \tilde{B}(\phi)\,(\boldsymbol{u}-\boldsymbol{v}) - p_e \nabla\phi. \tag{2.3e}$$

The first term represents the drag force, the second results from voidage gradients in the fluidized bed. The drag coefficient should be chosen such that for small values of $(1-\phi)$ the Stokes law for a single sphere moving through a viscous medium is obtained, i.e.

$$\tilde{B}(\phi) = \frac{1-\phi}{\phi^n}\frac{D_0}{V_p} \tag{2.3f}$$

with $n \approx 3$ and

$$\frac{D_0}{V_p} = \frac{\text{Stokes' drag on a single particle}}{\text{Volume of a particle}} = \frac{9}{2}\frac{\mu_f}{r^2}, \tag{2.3g}$$

where r denotes the radius of the particles.

We emphasize that the precise form of the drag coefficient is not crucial for our analysis; the above form was chosen just for definiteness. More complex forms, in particular an additional dependency on $|\boldsymbol{u}-\boldsymbol{v}|$, may also be taken into account. Other interaction forces are present, e.g. the so-called virtual mass effect, the precise form of which is also not known. Since it turns out that a virtual mass term would have a minor influence on the stability properties of both gas and liquid fluidized beds only, it will not be taken into account here.

2. *The basic state*

As basic state we denote that stationary and homogeneous solution of (2.3*a–e*), for which the particles do not move, while the fluid is moving with constant velocity against the direction of gravity. This state is unique up to Galilei transformations, if either the corresponding voidage ϕ_0 or the fluidization velocity $\boldsymbol{u}_0$ is given; in the latter case the function $\phi(1-\phi)/\tilde{B}(\phi)$ must be monotone in the interval $I := (0,1)$. The following relations apply:

$$\boldsymbol{u}_0 = \frac{\phi_0(1-\phi_0)}{\tilde{B}(\phi_0)}(\rho_s - \rho_f)g\boldsymbol{k}, \tag{2.4a}$$

$$\nabla p_e^0 = -[(1-\phi_0)\rho_s + \phi_0\rho_f]g\boldsymbol{k}. \tag{2.4b}$$

Here $\boldsymbol{k} = -\boldsymbol{g}/|\boldsymbol{g}|$ represents the unit vector in the vertical direction which will be denoted by x; y is the horizontal direction and $\boldsymbol{x}$ the vector $(0, y, x)$.

From now on, the index e denoting the effective fluid pressure p_e will be omitted, since the particle phase pressure has been eliminated by introducing the function $g(\phi)$.

Let $\delta \equiv \rho_f/\rho_s < 1$. For the special $B(\phi)$ of (2.3*f, g*), the following 'compatibility relation' for the diverse parameters results from (2.4*a*):

$$\frac{9}{2}\frac{\mu_f}{r^2}u_0 = (1-\delta)\rho_s g\phi_0^{n+1}, \tag{2.5}$$

where $u_0 = |\boldsymbol{u}_0|$.

2.3. *Rescaling*

The characteristic values, which occur in the basic state (2.4), are used for writing (2.3) in dimensionless form. Therefore, let us define the new variables:

$$\left.\begin{aligned} \boldsymbol{x} &=: r\hat{\boldsymbol{x}} \\ (\boldsymbol{u}, \boldsymbol{v}) &=: u_0(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) \\ (p, g) &=: \rho_s g r(\hat{p}, \hat{g}), \end{aligned}\right\} \tag{2.6a}$$

and

$$t =: \frac{r}{u_0}\hat{t}, \tag{2.6b}$$

$$\tilde{B} =: \frac{\rho_s g}{u_0}\hat{B} = \frac{\tilde{B}(\phi_0)}{\phi_0(1-\phi_0)(1-\delta)}\hat{B}. \tag{2.6c}$$

This leads to the following parameters:

$$\left.\begin{array}{ll} F = u_0^2/gr & \text{Froude number,} \\ R = \rho_s u_0 r/\mu_s & \text{Reynolds number,} \\ \delta = \rho_f/\rho_s (< 1) & \text{density ratio,} \\ \kappa = \dfrac{\lambda_s + \frac{1}{3}\mu_s}{\mu_s}, \quad \bar{\kappa} = \dfrac{\lambda_f + \frac{1}{3}\mu_f}{\mu_f}, \quad \nu = \mu_f/\mu_s & \text{viscosity coefficients.} \end{array}\right\} \tag{2.6d}$$

λ_α and μ_α (α = s or f) are the volume and shear viscosity coefficients. For our purposes it is sufficient to keep the viscosity coefficients constant.

In addition we define

$$\nabla[(1-\phi)\,g(\phi)] =: G(\phi)\,\nabla\phi, \tag{2.7}$$

where according to (2.2*c*),

$$G(\phi) = -g(\phi)+(1-\phi)\,g'(\phi) < 0 \quad \text{for all} \quad \phi\in(0,1). \tag{2.8}$$

Thus, the following system of equations is obtained from (2.3*a–g*) together with (2.2*a, b*), where the 'hat' is omitted for convenience:

$$-\dot{\phi}+\nabla\cdot[(1-\phi)\,\boldsymbol{v}] = 0, \tag{2.9a}$$

$$\dot{\phi}+\nabla\cdot(\phi\boldsymbol{u}) = 0, \tag{2.9b}$$

$$F(1-\phi)\,(\dot{\boldsymbol{v}}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}) = -(1-\phi)\,\boldsymbol{k}+B(\phi)\,(\boldsymbol{u}-\boldsymbol{v})-G(\phi)\,\nabla\phi$$
$$-(1-\phi)\,\nabla p+\frac{F}{R}(\nabla^2+\kappa\nabla\nabla\cdot)\boldsymbol{v}, \tag{2.9c}$$

$$F\delta\phi(\dot{\boldsymbol{u}}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\delta\phi\boldsymbol{k}-B(\phi)\,(\boldsymbol{u}-\boldsymbol{v})-\phi\nabla p+\nu\frac{F}{R}(\nabla^2+\bar{\kappa}\nabla\nabla\cdot)\,\boldsymbol{u}. \tag{2.9d}$$

$$B(\phi) = \frac{\tilde{B}(\phi)}{\tilde{B}(\phi_0)}\phi_0(1-\phi_0)\,(1-\delta) = (1-\delta)\frac{1-\phi}{\phi^n}\phi_0^{n+1} \quad \text{(drag force)}, \tag{2.9e}$$

$$\phi_0^{n+1} = \frac{1}{1-\delta}\tfrac{9}{2}\nu\frac{F}{R} \quad \text{(compatibility relation)}; \tag{2.9f}$$

Periodic boundary conditions in x and y. (2.9*g*)

$$\boldsymbol{k} = -\boldsymbol{g}/|\boldsymbol{g}|, \quad \boldsymbol{u} = (0, u_y, u_x). \tag{2.9h}$$

According to (2.4*a, b*) and the normalization (2.6), the basic state now reads

$$\phi = \phi_0, \quad \boldsymbol{v} = 0, \tag{2.10a}$$

$$\boldsymbol{u}_0 = \frac{\phi_0(1-\phi_0)}{B(\phi_0)}(1-\delta)\,\boldsymbol{k} = \boldsymbol{k}, \tag{2.10b}$$

$$\nabla p_0 = -[1-\phi_0(1-\delta)]\,\boldsymbol{k}. \tag{2.10c}$$

It must be noted that the Froude number F still depends on ϕ_0. With $B(\phi)$ of (2.9*e*) it follows that

$$F = \frac{u_0^2}{gr} = \frac{u_t^2\,\phi_0^{2(n+1)}}{gr}, \tag{2.11}$$

where

$$u_t = (1-\delta)\frac{\rho_s g}{\frac{9}{2}\mu_f/r^2} \tag{2.12}$$

is the terminal free-fall velocity of a single particle in the stationary viscous fluid for $\delta \ll 1$. The relation $u_0 = u_t\phi_0^{n+1}$ (here we choose $n = 3$) is Richardson's correlation for the expansion of uniformly fluidized beds (Richardson 1971).

A two-dimensional fluidized bed in the (y, x)-plane will be studied, where x denotes the vertical direction (against gravity) and y denotes the horizontal coordinate axis.

In order to keep the eigenvalues of the linearized equations discrete and the linear stability analysis as simple as possible, periodic boundary conditions are chosen. This procedure is also suggested by the results in one space dimension, where the existence of travelling waves oscillating in space as well as in time can be shown (Needham & Merkin 1986). These results can then be extended in various directions, namely to global results and two dimensions (Göz 1990*b*).

In most investigations so far the fluid has been considered an ideal gas, i.e. ν (and $\nu\bar{\kappa}$) has been set to zero. More precisely, 'only' the dissipation terms of the fluid flow have been neglected, but of course not the interaction term $B(\phi)$, which is proportional to ν according to the compatibility relation (2.9*f*). Besides the change of type of equation (2.9*d*), this is a virtually inconsistent procedure but might be justified by asymptotic analysis considerations. The influence of the fluid dissipation on possible bifurcations will be evaluated in §4.

The additional simplification $\delta = 0$ is proposed quite often, since in the 'interesting' cases, in which bubbles occur, it has been observed that $\rho_f \ll \rho_s$, i.e. $\delta \ll 1$. Neglecting the terms $\sim \delta$ of course changes (2.9*d*) drastically, and even more so when $\nu = 0$ (in the above sense) is assumed in addition. On the other hand, this is convenient, since the fluid velocity can then be eliminated algebraically. For the travelling waves this has the consequence that a branch of periodic solutions bifurcation from the basic state must terminate (almost) necessarily in a solution of infinite period, whereas for certain $\delta \neq 0$ it can return to the basic state. This may restabilize the basic solution (Göz 1990*a*, *b*).

For reference we formulate the reduced model, which is obtained by setting $\delta = 0$ and neglecting the viscosity of the light phase ($\nu = 0$ in the sense discussed above). We shall not discuss the validity of this singular limit approach.

Then (2.9*d*) becomes

$$0 = -B(\phi)\,(\boldsymbol{u}-\boldsymbol{v}) - \phi\nabla p. \tag{2.13}$$

Moreover, if we add (2.9*a*) and (2.9*b*) to eliminate $\partial_t \phi$ from one of these equations, the resulting equation expresses the fact that the mean flow $[(1-\phi)\,\boldsymbol{v}+\phi\boldsymbol{u}]$ is divergence-free. Solving now (2.13) for $\boldsymbol{u}$ and replacing $\boldsymbol{u}$ in the other equations gives

$$-\partial_t \phi + \nabla\cdot[(1-\phi)\,\boldsymbol{v}] = 0, \tag{2.14a}$$

$$\nabla\cdot\left(\boldsymbol{v} - \frac{\phi^2}{B(\phi)}\nabla p\right) = 0, \tag{2.14b}$$

$$F(1-\phi)\,(\dot{\boldsymbol{v}} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = -(1-\phi)\,\boldsymbol{k} - \nabla p - G(\phi)\,\nabla\phi + \frac{F}{R}(\nabla^2 + \kappa\nabla\nabla\cdot)\,\boldsymbol{v}. \tag{2.14c}$$

Taking $(1-\phi)$ as the relevant variable leads to the usual one-phase barotropic compressible fluid flow equations with the external force being replaced by a pressure-gradient term, which is in turn a function of the flow itself.

3. Linear stability analysis of the basic state

3.1. *Derivation of a linear equation for the voidage; definition of parameters*

We describe briefly the results of the linear stability analysis of the basic state. In addition, we introduce several important parameters and prove some relations among them.

The linearization of (2.9) in $(\phi, \boldsymbol{u}, \boldsymbol{v}, \nabla p)_0 = (\phi_0, \boldsymbol{k}, \boldsymbol{0}, \partial_x p_0 \boldsymbol{k})$ is given by

$$-\dot{\phi} + (1-\phi_0)\nabla\cdot\boldsymbol{v} = 0, \tag{3.1a}$$

$$\dot{\phi} + \phi_0 \nabla\cdot\boldsymbol{u} + \partial_x \phi = 0, \tag{3.1b}$$

$$F(1-\phi_0)\dot{\boldsymbol{v}} = (1+B_0'+\partial_x p_0)\phi\boldsymbol{k} + B_0(\boldsymbol{u}-\boldsymbol{v}) - G_0\nabla\phi - (1-\phi_0)\nabla p + \frac{F}{R}(\nabla^2+\kappa\nabla\nabla\cdot)\boldsymbol{v}, \tag{3.1c}$$

$$F\delta\phi_0(\dot{\boldsymbol{u}}+\partial_x\boldsymbol{u}) = -(\delta+B_0'+\partial_x p_0)\phi\boldsymbol{k} - B_0(\boldsymbol{u}-\boldsymbol{v}) - \phi_0\nabla p + \nu\frac{F}{R}(\nabla^2+\bar{\kappa}\nabla\nabla\cdot)\boldsymbol{u}; \tag{3.1d}$$

$$\text{periodic boundary conditions for } (\phi, \boldsymbol{u}, \boldsymbol{v}, \nabla p). \tag{3.1e}$$

Here $G_0 = G(\phi_0)$, $B_0 = B(\phi_0)$, $B_0' = \partial_\phi B(\phi_0)$.

From these equations the following scalar equation can be derived for ϕ alone:

$$(A\partial_t^2 + 2C\partial_t\partial_x + C\partial_x^2 + D\partial_x + E\partial_t - M\nabla^2 - J\partial_t\nabla^2 - H\partial_x\nabla^2)\phi = 0, \tag{3.2}$$

where the positive constants are given by

$$\left.\begin{aligned}
A &= \phi_0 + C, \quad C = \delta(1-\phi_0),\\
D &= \left[\frac{B_0}{\phi_0} - B_0' + (1-\delta)(1-2\phi_0)\right]\Big/F = (n+2)(1-\phi_0)(1-\delta)/F,\\
E &= \frac{B_0}{F\phi_0(1-\phi_0)} = \frac{1-\delta}{F}, \quad M = \frac{\phi_0}{F}|G_0|,\\
H &= \frac{\nu}{R}\frac{(1+\bar{\kappa})(1-\phi_0)}{\phi_0}, \quad J = \frac{(1+\kappa)\phi_0}{R(1-\phi_0)} + H.
\end{aligned}\right\} \tag{3.3}$$

The coefficient A arises essentially through particle inertia, C is associated with the inertia of the fluid, D with the drag, E contains the gravity force, M is due to the contact stress, and H and J are correlated to viscous dissipation (cf. Homsy 1983; Liu 1982).

To proceed let us define some combined parameters and a function f, which will be crucial for the subsequent stability and bifurcation investigations, respectively.

$$m := \frac{M}{A}, \quad c := \frac{C}{A}, \quad d := \frac{D}{E}, \quad h := \frac{H}{J}. \tag{3.4}$$

$$f(s) := m - c(1-c) - (s-c)^2. \tag{3.5}$$

We state some simple but important properties of these coefficients and of f. It is easy to see that for all $\phi_0 \in (0,1)$, $\delta \in [0,1)$, $\nu \in [0,1)$ the following relations hold:

$$0 \leqslant c < 1-\phi_0 < 1; \quad c < \frac{d}{n+2}; \quad 0 \leqslant h < 1; \tag{3.6}$$

and for physical reasons

$$h < d. \tag{3.7}$$

Then it follows also that

$$f(h) > f(d). \tag{3.8}$$

Furthermore, the function $f(s)$ takes its maximum at $s = c$ with $f(c) = m - c(1-c)$, and satisfies $f(s) < m$ for all $(s, c) \neq (0,0)$.

According to (3.3) and (3.4), we have

$$c = \frac{C}{A} = \frac{\delta}{\delta(1-\phi_0)+\phi_0}(1-\phi_0) < 1-\phi_0 < 1,$$

and

$$c < 1-\phi_0 = \frac{d}{n+2}.$$

Also, it is clear that $h < 1$ since $H < J$. Now assume $h \geqslant d$ (of course this is only possible if $\nu \neq 0$). Then, because $h < 1$ we also have $d = (n+2)(1-\phi_0) < 1$ and this means that $1 > \phi_0 > (n+1)/(n+2)$. Writing out the assumption gives

$$(n+2)(1-\phi_0) \leqslant h = \frac{H}{J} = \frac{\nu(1+\bar{\kappa})(1-\phi_0)}{\nu(1+\bar{\kappa})(1-\phi_0)+(1+\kappa)/(1-\phi_0)\phi_0^2} < 1,$$

and hence the following estimate holds:

$$\frac{\nu(+\bar{\kappa})}{n+2} > \nu(1+\bar{\kappa})(1-\phi_0) \geqslant (n+2)(1-\phi_0)\left[\nu(1+\bar{\kappa})(1-\phi_0)+\frac{1+\kappa}{1-\phi_0}\phi_0^2\right]$$

$$> (n+2)(1+\kappa)\phi_0^2 > (1+\kappa)\frac{(n+1)^2}{n+2}.$$

Therefore

$$\nu\frac{1+\bar{\kappa}}{1+\kappa} > (n+1)^2 \quad \text{or} \quad \frac{\lambda_f+\frac{4}{3}\mu_f}{\lambda_s+\frac{4}{3}\mu_s} > (n+1)^2 \approx 16.$$

However, the left-hand side is expected to be $\ll 1$, since in general for example $\mu_f/\mu_s \sim O(10^{-1})$ (F. Ebert 1987, personal communication), and similarly $\lambda_f \ll \lambda_s$. Hence the physical parameters are chosen such that we can confine our investigations to the case of $d > h$.

The inequality (3.8) follows from (3.6) and (3.7):

$$f(h)-f(d) = (d-h)(d+h-2c) > (d-h)\left(\frac{n}{n+2}d+h\right) > 0.$$

Finally, the properties of f are obvious from $0 \leqslant c < 1$, where equality holds for $\delta = 0$.

3.2. *The dispersion relation, stability conditions*

Stability results are obtained using the ansatz $\phi = \varphi \exp(\sigma t + i\lambda x + iky)$. This gives the dispersion relation

$$A\sigma^2+[E+2iC\lambda+J(\lambda^2+k^2)]\sigma+iD\lambda-C\lambda^2+M(\lambda^2+k^2)+iH\lambda(\lambda^2+k^2) = 0, \quad (3.9)$$

which has to be investigated for $\sigma \in \boldsymbol{C}$, $(\lambda, k) \in \boldsymbol{R}^2\backslash\{(0,0)\}$. (The solutions $\phi \equiv$ const. or $\lambda = k = 0$ are trivial.)

Note that the correct spectral ansatz $(\phi, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\nabla} p) = \boldsymbol{c}\exp(\sigma t + i\lambda x + iky)$ yields the following result – after a large amount of algebra: the expression on the left-hand side of (3.9) is multiplied by $(\lambda^2+k^2)\tilde{q}$, where $\tilde{q}$ is a function of σ, λ, and k^2. Setting $\lambda = k = 0$ leads trivially to $\sigma\varphi = 0$, whereas the zeros of $\tilde{q}$ are located in the negative half-plane ($\mathrm{Re}\,\sigma < 0$) and thus yield decaying modes only. Therefore, it is allowable to restrict the analysis to the single equation (3.2).

Separating σ into real and imaginary parts with $\sigma = \sigma_r + i\sigma_i$, the relation

$$\sigma_i \sim -\lambda \quad (3.10)$$

is obtained, while σ_r is determined by the two real zeros of a fourth-order polynomial. By elementary calculations we obtain the following criterion for *conditional stability*: the basic state is stable (in the linearized sense) against perturbations with wave numbers $(\lambda, k) \in \boldsymbol{R}^2$, for which the following relation holds:

$$\lambda^2 f(q(\lambda^2+k^2)) + mk^2 > 0, \quad q(s) := \frac{D+Hs}{E+Js}. \tag{3.11}$$

From this the stability condition for *all* (λ, k) follows (*unconditional stability*): there are no growing modes, if the following relation is satisfied:

$$f(d) \geqslant 0. \tag{3.12}$$

To show this we first evaluate the condition (3.11) at $k = 0$ which leads to the condition that for $q(\lambda^2) > 0, f(q) = m - c + 2cq - q^2 > 0$ must also hold. Since

$$\frac{\mathrm{d}}{\mathrm{d}\lambda^2} q(\lambda^2) = \frac{JE(h-d)}{(E+K\lambda^2)^2} < 0$$

in accordance with (3.7), and since $q(0) = d$, $q(\infty) = h$, it follows that $q \in (h, d)$. Because $f(q)$ represents a parabola open to the lower half-space and has to be positive in the interval (h, d), the two relations $f(h) \geqslant 0$, $f(d) \geqslant 0$ have to be satisfied. But according to (3.8), it is sufficient to require $f(d) \geqslant 0$.

Then it is shown that (3.12) is also sufficient for $k \neq 0$: because $f(q(s))$ can also be written in the form

$$f(q(s)) = \frac{f(h)\, J^2 s^2 + f(d)\, E^2 + [f(h) + f(d) + (d-h)^2]\, EJs}{(E+Js)^2},$$

it is obvious that $f(q(\lambda^2+k^2)) > 0$, if $f(d) \geqslant 0$. Hence, (3.11) is satisfied.

From the above considerations we can note the well-known fact that the main instability lies in the vertical direction. There are further (transversal) instabilities present in the system, but these are more convenient to investigate within the travelling wave approach of the next chapter.

3.3. *Remarks*

(i) As was already noted by Needham & Merkin (1983), the porosity gradient term proportional to M is decisive for the stability of the homogeneous fluidized bed. In the subsequent investigation of the travelling waves this coefficient will be seen to determine mainly their velocity. Although the viscosity coefficients do not enter the stability condition, their presence (but not their values) is decisive for the bifurcation to periodic travelling waves.

(ii) In terms of the physical parameters, condition (3.12) means

$$|G_0| \geqslant Fd^2 + \delta F(1-d)^2(1-\phi_0)/\phi_0, \tag{3.13}$$

whereas $f(d)|_{\delta=0} = m - d^2 \geqslant 0$ corresponds to the well known version of

$$|G_0| \geqslant Fd^2 = F(n+2)^2(1-\phi_0)^2, \tag{3.14}$$

taking into account (2.10). (We recall that often $G(\phi) \equiv$ const. is assumed.) Thus, when $\delta \neq 0$ additional restrictions on the stability of the basic state are obtained.

(iii) A closer look at the stability conditions (3.12), (3.13) reveals the following tendencies. The stability of the basic state decreases with increasing particle inertia or decreasing contact stress; it also decreases with increasing drag. The role of the fluid inertia depends on the other parameters, but is seen to have a mainly stabilizing effect: a high fluid inertia is destabilizing, if $d < \frac{1}{2}$, and stabilizing otherwise. Now $d < \frac{1}{2}$ means $\phi_0 > (2n+3)/(2n+4) \doteq 0.9$ for the particular coefficients considered here, and this shows the stabilizing influence of the fluid inertia in most regions of the bed.

In addition, the stability of the homogeneous state is restricted by an increasing ratio δ of the specific densities of the two phases and by an increasing Froude number, i.e. an increasing fluidization velocity or a smaller particle size.

4. Possible bifurcations of two-dimensional travelling waves

4.1. *Survey*

We are searching for travelling wave solutions of (2.9), i.e. solutions of the form

$$U(x,y,t) = \hat{U}(x-\omega t, y). \tag{4.1}$$

These waves are travelling mainly in the vertical direction ('downstream') and possibly have a transverse structure. Because of our boundary conditions this means periodicity in y. We want to find out for which 'wave speeds' ω the solution (if it exists) is also periodic in

$$z := x-\omega t, \quad \omega = \text{const.} \tag{4.2}$$

Below we analyse the linearized equations in the moving coordinate system in order to obtain information about the possible bifurcation points. Here ω is considered the bifurcation parameter. The validity of this procedure and the actual bifurcation to travelling waves of the above form has been shown partially (Göz 1990*b*) by use of the Lagrangian formulation of a small modification of the reduced model, which is obtained by setting $\delta = \nu = 0$ (cf. (2.14)). We shall return to this point at the end, but assume for the time being that the conclusions which might be drawn from linear theory transfer to the nonlinear regime. Nevertheless, special travelling wave solutions can be obtained for the full equations by the more restrictive ansatz (Göz 1990*a*, *b*)

$$U(x,y,t) = \tilde{U}(x-\omega t \pm \tilde{k}y). \tag{4.3}$$

We will find several double-periodic solutions of the transformed equations linearized in the basic state depending on the wavenumber k in the y-direction. Because of periodicity, k is restricted to

$$k = \frac{2\pi l}{\alpha}, \quad l \in N^0, \quad \alpha = \text{bed width (in units of the lengthscale } r). \tag{4.4}$$

Now the original system of equations as well as the basic state possesses the symmetries $O(2)_y \times SO(2)_x \times S_t^1$. These are due to the invariance under translations in space and time – which are identified with rotational symmetries by the demand of periodicity – and a reflectional symmetry in the horizontal direction. The identification of the action of $SO(2)_x$ with that of S_t^1 is found to be equivalent to the travelling wave ansatz (cf. Golubitsky & Stewart 1986 for a similar problem, namely the Taylor–Couette flow). This yields the spatial symmetry $O(2)_y \times SO(2)_z$, and on the basis of the well-known arguments with regard to bifurcation with symmetry (e.g. Chossat & Iooss 1985; Erneux & Matkowsky 1984; Golubitsky & Stewart 1985; Iooss 1984; Vanderbauwhede 1982), both travelling and standing (travelling) waves are expected.

More precisely, the pure 'travelling waves' (TW) are of the form $U(x-\omega t \pm \tilde{k}y)$, where $\tilde{k}$ is correlated to the periodicity in the y-coordinate (see (4.17) below), while the 'standing travelling waves' (STW) are of the form $U(x-\omega t,\ ky)$. When $\tilde{k} = 0$ ($\hat{=}$ k $= 0$), we have a vertical travelling plane wave train (VTW), which we call a one-dimensional TW; when $\tilde{k} \neq 0$, the waves are planar, but actually two-dimensional and are called oblique travelling waves (OTW).

Of course the STWs with $k \neq 0$ are also travelling (in the x-direction) waves, but with a non-trivial transverse structure. Therefore, they are referred to as standing (with respect to the y-direction) waves. They are mainly two-dimensional and coupled to the OTWs by symmetry. In fact they grow out of the same point of bifurcation and can locally (i.e. in the linearized sense) be considered a superposition of two oblique waves moving both downstream but at a skew angle. Additionally it follows that the OTW and STW may interact, as a result of which quasi-periodic solutions are obtained. For this fact see the references mentioned above and the recent paper by Bridges (1989), who studied the bifurcation from one-phase plane Poiseuille flow and obtained similar results.

We shall not deal with the computations in detail, since it is at least known in principle what can happen in such cases. In particular, a detailed stability analysis would require numerical methods, as the analytical computations are almost impossible to carry out. Here and in the next section we shall concentrate on the main features of the possible bifurcations, whereas more detailed investigations, especially of the OTWs, will be presented in other publications (for a survey see Göz 1990*a*).

Instead of looking for Hopf bifurcations to time-periodic solutions of the original system, it is considered more convenient to transform the equations to a coordinate system moving with the constant velocity ω and to look there for temporally stationary, spatially periodic solutions as a function of the new parameter ω. The main reason for doing so is that the eigenvalues σ of the original system are not complex conjugated for $\delta \neq 0$, as can be seen from (3.3). Thus, it would be necessary anyway to perform a Galilei transformation with $\omega = c$. Of course, the bifurcation approach is conceptionally different, but the outcome is the same ($-\omega$ merely replaces σ). This relies on the fact that the symmetries $SO(2)_x$ and S^1_t are identified by the travelling wave ansatz (4.1) – which is justified by the fact that $\sigma_i \sim -\lambda$, see (3.10). Thus, we can look equally well for the (Hopf) bifurcation of a solution, which is periodic in z.

With the substitutions $\partial_t = -\omega\partial_z$, $\partial_x = \partial z$ following from the definition above, (2.9) become an elliptic system of mixed order:

$$\nabla\cdot[(1-\phi)(\boldsymbol{v}-\omega\boldsymbol{k})] = 0, \tag{4.5a}$$

$$\nabla\cdot[\phi(\boldsymbol{u}-\omega\boldsymbol{k})] = 0, \tag{4.5b}$$

$$F(1-\phi)(\boldsymbol{v}-\omega\boldsymbol{k})\cdot\nabla\boldsymbol{v} = -(1-\phi)\boldsymbol{k}+B(\phi)(\boldsymbol{u}-\boldsymbol{v})-G(\phi)\nabla\phi -(1-\phi)\nabla p+\frac{F}{R}(\nabla^2+\kappa\nabla\nabla\cdot)\boldsymbol{v}, \tag{4.5c}$$

$$F\delta\phi(\boldsymbol{u}-\omega\boldsymbol{k})\cdot\nabla\boldsymbol{u} = -\delta\phi\boldsymbol{k}-B(\phi)(\boldsymbol{u}-\boldsymbol{v})-\phi\nabla p+\nu\frac{F}{R}(\nabla^2-\bar{\kappa}\nabla\nabla\cdot)\boldsymbol{u}; \tag{4.5d}$$

periodic boundary conditions. (4.5*e*)

Here the spatial derivatives refer to y and z, and $\boldsymbol{k} = (0,1)$.

4.2. *Linear theory*

The possible bifurcating solutions follow from the kernel of the operator corresponding to (4.5) and linearized in the basic state (2.5). Proceeding as at the beginning of the last section gives a scalar equation for the voidage:

$$[(A\omega^2-2C\omega+C-M)\,\partial_z^2+(D-E\omega)\,\partial_z-M\partial_y^2+(J\omega-H)\,(\partial_z^2+\partial_y^2)\,\partial_z]\,\phi=0. \quad (4.6)$$

As before, this procedure is not necessary, but convenient. The detailed analysis (evaluation of a 6×6 determinant) yields the left-hand side of (4.8) below times a non-vanishing prefactor. Thus, it is sufficient to restrict the investigation to the solutions of (4.6).

The ansatz $\phi=\varphi_1(z)\varphi_2(y)$ yields

$$\varphi_1(z)=\mathrm{e}^{\Lambda z}, \quad \varphi_2(y)=\varphi_2^+\,\mathrm{e}^{\mathrm{i}ky}+\varphi_2^-\,\mathrm{e}^{-\mathrm{i}ky}, \quad (4.7)$$

with
$$J(\omega-h)\,\Lambda^3-Af(\omega)\,\Lambda^2+[E(d-\omega)-J(\omega-h)\,k^2]\,\Lambda+Mk^2=0, \quad (4.8)$$

where the definitions (3.4) and (3.5) are taken into account. Note that Λ depends only on k^2 by symmetry. According to (4.5*e*), $\Lambda=\mathrm{i}\lambda$, $\lambda\in\boldsymbol{R}$ such that the two real equations

$$[E(d-\omega)-J(\omega-h)\,(k^2+\lambda^2)]\,\lambda=0, \quad (4.9a)$$

$$f(\omega)\,\lambda^2+mk^2=0 \quad (4.9b)$$

are obtained. Considering the propagation velocity ω the bifurcation parameter, our preliminary observations can be summarized as follows.

(i) The possible bifurcation points $(\lambda,\omega,k)_c$ result from (4.9*a*, *b*). The eigenvalues λ, i.e. the wavenumbers in the vertical direction, depend on ω and k as $\lambda^2=\lambda^2(\omega(k^2))$, where the horizontal wavenumber k assumes discrete values according to the periodic boundary conditions (cf. (4.4)).

(ii) Because of symmetry, $\lambda=0$ is a trivial solution of (4.9*a*) giving no rise to a bifurcation.

(iii) For $\omega=d$ (4.9*a*) gives $\lambda^2+k^2=0$. Therefore, a stationary solution, which then is trivially symmetric, could bifurcate from the basic state at this parameter value. This branch and its interaction with the periodic solutions was studied by Göz (1990*a*, *b*) for OTWs using the ansatz (4.3). It gives rise to a codimension-2 bifurcation and homoclinic (infinite periodic) solutions.

(iv) For $\lambda\neq 0$, it can be concluded directly from (4.9*a*) and (4.9*b*), respectively, that bifurcation points can only exist if the critical values of ω satisfy the two relations

$$h<\omega<d \quad \text{and} \quad f(\omega)=m-c+2c\omega-w^2\leqslant 0 \quad (4.10)$$

since $h<d$ according to (3.7) and $m>0$. But then the stability condition (3.12) is violated so that a bifurcation to periodic travelling waves is expected to occur only when the basic state is unstable. This is also seen by comparing (4.9*a*, *b*) with (3.11).

To analyse (4.9), we first consider λ a function of the bifurcation parameter ω. This shows that in the domain determined by (4.10) λ^2 is well defined.

The function

$$\lambda^2=\lambda^2(\omega)=\frac{E}{J}\frac{d-\omega}{\omega-h}\frac{m}{m-f(\omega)} \quad (4.11)$$

is positive for all $\omega\in(h,d)$ and strictly monotonically decreasing, if $c\leqslant h$. If $c>h$, $\lambda^2(\omega)$

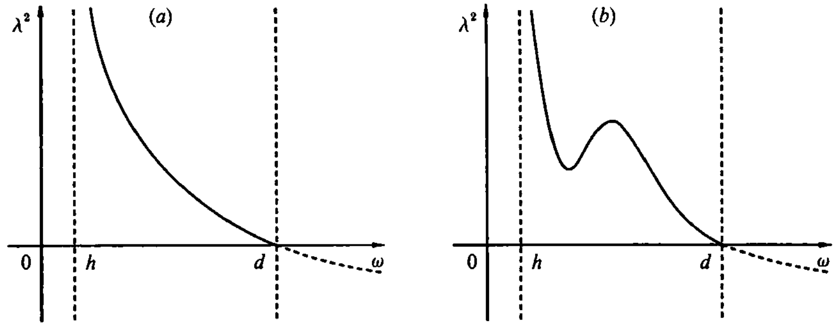

FIGURE 2. Critical longitudinal wavelength *vs.* travelling wave velocity: (*a*) $\lambda^2(\omega)$ for $c \leqslant h$; (*b*) possible behaviour for $c > h$.

is strictly monotonically decreasing in $(h, \frac{1}{3}(c+2h)] \cup [c, d)$. *If* $\omega \in (\frac{1}{3}(c+2h), c)$, *a parameter range exists in which* $\lambda^2(\omega)$ *is strictly monotonically increasing. A necessary and sufficient condition for the latter is*

$$(c-h)^2(9d-8c-h)-27(d-h)\,c(1-c) = 9(d-h)\,[(c-h)^2-3c(1-c)]-8(c-h)^3 > 0,$$

which also means $\phi_0 < \delta/(3+\delta)-O(h)$.

The last inequality shows that ϕ_0 has to be very small for a non-monotonic behaviour of $\lambda^2(\omega)$. Typical graphs of $\lambda^2(\omega)$ are sketched in figure 2. Roughly speaking, the longitudinal wavelength depends uniquely on the travelling wave velocity if the ratio of the inertia of the fluid to that of the particle phase does not exceed the corresponding ratio of the viscosity coefficients (modulated by a factor depending on the voidage value of the state of homogeneous fluidization). Otherwise ambiguities may occur, which will be discussed further in §5.

The above statement can be proved as follows. The eigenvalues are obtained as a function of ω using (4.9*a*, *b*) by eliminating k^2. The positivity of $\lambda^2(\omega)$ in (h, d) follows from the fact that $m-f(\omega) > 0$ for $\omega > h \geqslant 0$ (cf. §3). To answer the question for monotonicity, we have to consider the derivative of (4.9). This gives

$$Q(\omega) \equiv \frac{mJ}{E}\frac{\mathrm{d}}{\mathrm{d}\omega}\lambda^2(\omega) = \frac{Z}{N},$$

$$Z = -(\omega-h)\,(\omega^2-2c\omega+c)-(d-\omega)\,[\omega^2-2c\omega+c+2(\omega-h)\,(\omega-c)],$$

$$N = (\omega-h)^2(\omega^2-2c\omega+c)^2.$$

Note again that $\omega^2-2c\omega+c = m-f(\omega) > 0$ for $(\omega, c) \neq (0,0)$ and that $Q(h) = -\infty$, $Q(d) < 0$. For the following let $\omega \in (h, d)$. For $c \leqslant h$ it follows from direct inspection that $Q < 0$. Also $c > h$ gives $Q < 0$ for $\omega \geqslant c$. Now, the interval (h, c) remains to be considered. To do this, we first observe that $Z < 0$, if the term in the square brackets is positive, i.e. if $\omega < \frac{1}{3}(c+2h)$ 3. Next, we rewrite $Z(\omega)$ and take the derivatives:

$$Z(\omega) = 2\omega^3-(3d+2c+h)\,\omega^2+2d(2c+h)\,\omega-c(d-h+2hd),$$

$$Z' = 2[3\omega^2-(3d+2c+h)\,\omega+d(2c+h)] = 2(\omega-d)\,(\omega-\tfrac{1}{3}(2c+h)),$$

$$Z'' = 2[6\omega-(3d+2c+h)].$$

Hence $Z(d) < 0$, $Z'(d) = 0$, $Z''(d) > 0$; $Z'(\frac{1}{3}(2c+h)) = 0$, $Z''(\frac{1}{3}(2c+h)) < 0$; $Z(0) < 0$. So Z has a relative maximum in $\omega_{\mathrm{m}} = \frac{1}{3}(2c+h)$. Now, if $Z(\omega_{\mathrm{m}}) > 0$, i.e.

$$9(d-h)\,[(c-h)^2-3c(1-c)]-8(c-h)^3 > 0, \tag{4.12}$$

Z will be positive in a neighbourhood of ω_m. In particular, the term in the angular bracket of (4.12) will be positive, and this leads to

$$c > \tfrac{1}{8}(3+2h+[9+12h(1-h)]^{\frac{1}{2}}) > \tfrac{1}{4}(3+h).$$

(The negative sign of the root is excluded by the assumption of $c > h$.)

Let us first assume $h = 0$ in order to reduce (4.12) to

$$\tilde{Z}(c) := 36cd - 27d - 8c^2.$$

Then, $\tilde{Z}(\frac{3}{4}) < 0$, $\tilde{Z}(1) \sim 9d-8 > 0$, if $d > \frac{8}{9}$. But the latter can fulfilled, since from $c > \frac{3}{4}$ and $d > (n+2)\,c$ (see (3.6)) also $d > \frac{3}{4}(n+2) \approx \frac{15}{4} > \frac{8}{9}$ is obtained. (For $d \geqslant \frac{15}{8}$ there exists a $c_1 \in (\frac{3}{4}, 0.832]$ such that $\tilde{Z}(c_1) = 0$.) Furthermore, $c > \frac{3}{4}$ means that $\phi_0 < \delta/(3+\delta) < \frac{1}{4}$ applies, whereas from $d > \frac{3}{4}(n+2)$ only $\phi_0 < \frac{1}{4}$ is obtained. Owing to continuity this positivity assertion also applies to small $h \neq 0$, but with smaller ϕ_0 than stated.

Next, we investigate ω as a function of k (resp. k^2). For this, it is more convenient to consider conversely k^2 as a function of ω. The elimination of λ^2 from (4.9a, b) gives $k^2 = k^2(\omega)$ in the form

$$\frac{J}{E}k^2 = \frac{d-\omega}{\omega-h}\frac{-f(\omega)}{m-f(\omega)} =: P(\omega) \tag{4.13}$$

with the constraint (4.10) for $k^2 \geqslant 0$! We conclude that for given transverse wavenumber several periodic voidage waves travelling with different propagation velocities can exist:

For a given k the number of solutions $\omega \in (h, d)$ of (4.13) *lies between* 0 *and* 3. *For sufficiently large $|k|$ at most one solution exists. This depends on the relative magnitude of the parameters c, d, h and m.*

More precisely, the cases described below and illustrated in figure 3 can be distinguished (remember that $m - f(\omega) > 0$ for $\omega > 0$ and $\max f = f(c)$).

(I) No solution exists for $f(\omega) > 0$ in (h, d), i.e. $k^2 = P(\omega) < 0$ in (h, d).

(IIa) Exactly one solution exists in (h, d) if $k^2(\omega)$ is monotonically decreasing. This is true for $f(c) > 0$, $c < h - [f(c)]^{\frac{1}{2}}$, i.e. $f(h) < 0$; or $f(\omega) < 0$ for all ω, i.e. $f(c) < 0$.

(IIb) Generically one or three solutions exist if $k^2(\omega)$ is not monotonically decreasing. ($f(\omega) < 0$ in (h, d).)

(III) For small $k^2 \neq 0$, two solutions exist (generically), but not in the whole interval, if $c - [f(c)]^{\frac{1}{2}} < h < c + [f(c)]^{\frac{1}{2}} < d$, i.e. $f(d) < 0$, $f(h) > 0$.

(IV) For large k^2, exactly one solution exists, while for small k^2 three solutions exist (generically). This requires $h < c - [f(c)]^{\frac{1}{2}} < c + [f(c)]^{\frac{1}{2}} < d$, which means $c \in (h, d)$, $f(h) < 0$, $f(d) < 0$, $h + [f(c)]^{\frac{1}{2}} < c < d/(n+2)$.

(V) In the special case of $f(h) = 0$, one or two possible solutions are found for small k^2. Here we have

$$Jk^2/E = (d-\omega)(\omega+h-2c)/(m-f(\omega)), \quad h^2 - 2ch + c - m = 0.$$

(a) $h = c + [f(c)]^{\frac{1}{2}} > c$, with the subcases (i) $D_\omega k^2(h) > 0$, (ii) $D_\omega k^2(h) < 0$.

(b) $h = c - [f(c)]^{\frac{1}{2}} < c$.

The term 'generically' refers to each ω except for those for which $P(\omega)$ has an extremum.

To see this, we observe that P has zeros in $\omega = d$ and in $\omega_{1,2} = c \pm [f(c)]^{\frac{1}{2}}$, if $f(c) > 0$. Additionally, $P \to \pm\infty$ with $\omega \to h$, if $f(h) \neq 0$. When $f(h) = 0$ and $\omega = h$, we can factorize a factor $(\omega - h)$ in $f(\omega)$, which gives case (V). Otherwise there are seven cases regarding the positions of the zeros $\omega_{1,2}$ of f.

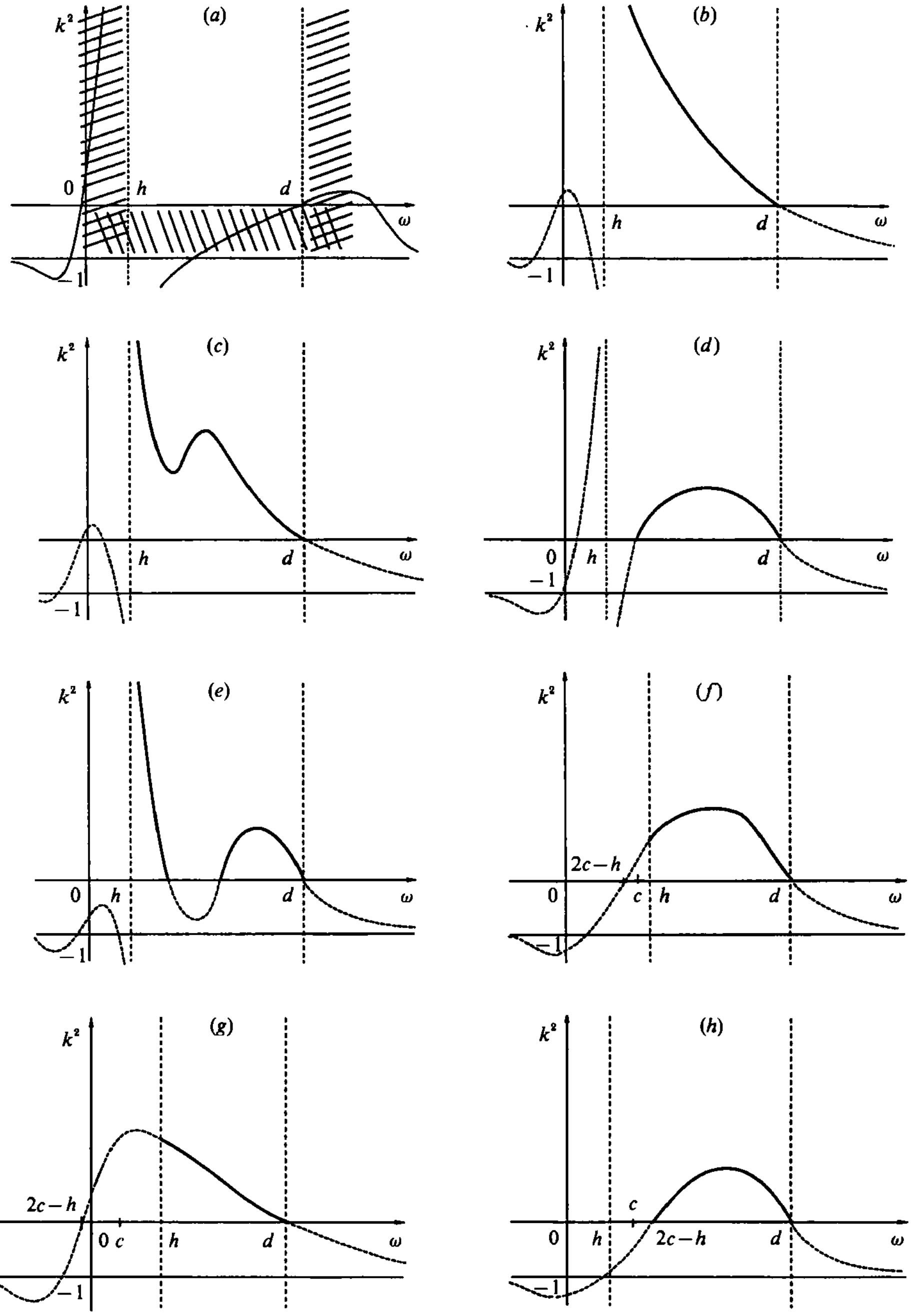

FIGURE 3. $k^2 = P(\omega)$ for different parameter values: (*a*) Case (I), (*b*) Case (II*a*), (*c*) Case (II*b*), (*d*) Case (III), (*e*) Case (IV), (*f*) Case (V*a*(i)), (*g*) Case (V*a*(ii)), (*h*) Case (V*b*).

(i) $f(\omega) < 0$ for all ω: this gives one of the cases named in (II*a*) and (II*b*) depending on the monotonicity of P.

(ii) $\omega_{1,2} < h$: this requires $c + [f(c)]^{\frac{1}{2}} < h$ and also leads to the cases (II*a*, *b*) because of $-f > 0$ for $\omega > h$.

(iii) $\omega_1 < h < \omega_2 < d$: obviously, $f(h) > 0$, $f(d) < 0$ so that a zero of P exists in (h, d). This gives case (III).

(iv) $\omega_1 < h < d \leqslant \omega_2$: this gives $f > 0$ in (h, d). Hence there is no positive k^2 which is case (I).

(v) $h < \omega_1 < w_2 < d$: the sign of P changes with increasing $\omega \in (h, d)$ from positive to negative and again to positive. This is case (IV).

(vi) $h < \omega_1 < d < \omega_2$: this is not possible here, since from the extended inequality

$$h < c-[f(c)]^{\frac{1}{2}} < \frac{d}{n+2} - [f(c)]^{\frac{1}{2}} < d < c+[f(c)]^{\frac{1}{2}} < \frac{d}{n+2} + [f(c)]^{\frac{1}{2}}$$

it follows that $[f(c)]^{\frac{1}{2}} > (n+1)/(n+2)$, and thus $0 < h < -dn/(n+2) < 0$, which is a contradiction.

(vii) $d < \omega_{1,2}$: This requires $d < c$ and hence contradicts (3.6).

In view of the reduced model (2.14), it is important to note that only cases (I) and (III) are possible for $h = c = 0$. If only the fluid dissipation is assumed to vanish ($h \sim \nu = 0$), the solutions (II*a*), (II*b*) and (V*a*) are cancelled. If the fluid inertia is neglected but not the fluid dissipation ($\delta \sim c = 0$, $h \neq 0$), solutions (IV) and (V*b*) drop out. It is interesting that in case (III) only a finite number of modes with $k = 2\pi l/\alpha$ can bifurcate.

Now we know the solution structure of (4.9*a*, *b*) and thus also the dimension of the kernel of the linearized operator at the points $(\phi_0;\ (\omega, \lambda, k)_c)$. For a proof of a bifurcation to periodic solutions, the following two results are still needed.

Let ω and $k \neq 0$ be given (for $k = 0$ see (ii) *above). If a pair of complex-conjugated eigenvalues exists, then the remaining eigenvalue is real and negative.*

This is easily seen by comparison of the ansatz $(\Lambda-(\beta+i\lambda))(\Lambda-(\beta-i\lambda))(\Lambda-\Lambda_3)$ with (4.8). This yields

$$\Lambda_3 + 2\beta = \frac{Af(\omega)}{J(\omega-h)},$$

$$2\Lambda_3\beta + \beta^2 + \lambda^2 = \frac{E(d-\omega)}{J(\omega-h)} - k^2,$$

$$\Lambda_3(\beta^2+\lambda^2) = -\frac{Mk^2}{J(\omega-h)}.$$

The last equality yields the desired result, cf. (4.10). (The remaining two equations give the already known conditions with regard to the existence of pure imaginary eigenvalues for $\beta = 0$.)

The complex-conjugated eigenvalues cross the imaginary axis transversally, i.e.

$$\frac{d}{d\omega}\operatorname{Re}\Lambda(\omega_c) \sim -\frac{d}{d\omega}k^2(\omega_c) \sim -P'(\omega_c) \neq 0 \tag{4.14}$$

generically, i.e. except for those values of ω_c for which the first derivative of P vanishes.

Taking the derivative of (4.8) with respect to ω in $\Lambda = i\lambda$ and noting (4.9*a*, *b*) yields the two real equations

$$2J(\omega-h)\,\lambda\operatorname{Re}\Lambda' - 2Af\operatorname{Im}\Lambda' - Af'\lambda = 0,$$

$$2Af\operatorname{Re}\Lambda' + 2J(\omega-h)\,\lambda\operatorname{Im}\Lambda' + [E + J(\Lambda^2+k^2)] = 0,$$

where f, f', and Λ' are evaluated at some critical value $\omega = \omega_c$ (here, the index c is omitted). From this it follows that

$$\frac{2}{AE}[A^2f^2+J^2(\omega-h)^2\lambda^2]\,\mathrm{Re}\,\Lambda' = \frac{mf'(d-\omega)}{m-f}-\frac{f(d-h)}{\omega-h} = -(m-f)(\omega-h)P'(\omega),$$

according to (4.13).

4.3. *Nonlinear theory*

Now it could be concluded that the bifurcation to periodic travelling waves takes place, because all the requirements of the linear theory are fulfilled. In particular, as an elliptic operator on a compact manifold the linear operator satisfies the Fredholm property (Choquet-Bruhat, DeWitt-Morette & Dillard-Bleick 1982; see also Grubb & Geymonat 1977). However, there is no link to the nonlinear operator, which has to be formulated as a differentiable operator in an appropriate function space. This space has to be chosen larger than required for the pure linear problem, because when differentiating the hyperbolic part of the equations regularity is lost. But in this space the range of the linear operator is not closed, hence, it is not a Fredholm operator. We return to this severe mathematical technical difficulty in the next section, where some resolution possibilities will be mentioned.

It is proven by Göz (1990*b*) for the case of $\delta = \nu = 0$ that a bifurcation to two-dimensional periodic solutions can indeed occur, but for a slightly modified model. In the general case one would like to draw the same conclusion by an application of the equivariant Lyapunov–Schmidt method leading to the equivariant bifurcation equations, i.e. the symmetry of the basic solution transfers to the bifurcation equations (Sattinger 1983; Vanderbauwhede 1982). We will not repeat the standard analysis of (Hopf) bifurcation in the presence of an $O(2)\times S^1$ symmetry ($SO(2)_z \approx S^1$), but instead refer to the literature cited above. From the considerations above and the other results (Göz 1990*a*, *b*), the following conclusions can be drawn:

Let ω be the bifurcation parameter. If for a given ω a pair of complex-conjugated pure imaginary eigenvalues $\Lambda_{1,2} = \pm i\lambda$ and a pair of wavenumbers $k = \pm(k^2)^{\frac{1}{2}}$ satisfying (4.9*a*, *b*) *exist, then it can be stated that*:

The eigenvalues are unique (they are semi-simple but double if $k \neq 0$ by symmetry), they cross the imaginary axis transversely almost anywhere, and the solution space of the linearization (4.6) *is two-dimensional or four-dimensional, if $k=0$ or $k \neq 0$, respectively. Corresponding to these eigenvalues the following bifurcating solutions are obtained*:

(i) dim ker = 2 ($k=0$): $U^0 = U(x-\omega t)$, *periodic in $z = x-\omega t$. This is a planar 'one-dimensional' wave travelling through the bed opposite to the direction of gravity* ($\omega > h > 0$).

(ii) dim ker = 4 ($k \neq 0$):

(*a*) $U_+ = U(x-\omega t+\tilde{k}y)$ *or* $U_- = U(x-\omega t-\tilde{k}y)$, *periodic in* $\tilde{z} = x-\omega t \pm \tilde{k}y$ (*resp.*), *i.e.*

$$U_{\pm} = \varphi\, \mathrm{e}^{\mathrm{i}\lambda(x-\omega t\pm \tilde{k}y)} + \text{c.c.} + \text{h.o.t.}, \tag{4.15}$$

where c.c. *means complex conjugation, and* h.o.t. *means higher-order terms. This is a pair of planar 'two-dimensional' waves travelling diagonally through the bed.*

They can also be considered counter-moving spiral waves on a cylinder mantle because of the periodicity in y. Moreover, all solutions exist on a torus owing to the periodicity in both space coordinates.

Furthermore, we assume the existence of the following solutions:

(ii) dim ker = 4 ($k \neq 0$):

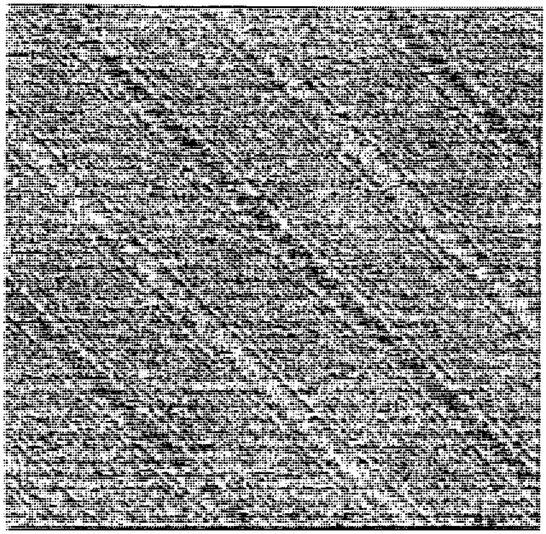

FIGURE 4. Contour plot of an oblique wave travelling diagonally to the right.

(*b*) [Proved in (Göz 1990*b*) for $\delta = \nu = 0$ and under an additional constraint, conjecture otherwise]

A wave with 'band structure' travelling downstream (opposite to the direction of gravity), but stationary in the spanwise direction:

$$U_k = U(x-\omega t, y) = (\varphi\, \mathrm{e}^{\mathrm{i}\lambda(x-\omega t)} + \mathrm{c.c.}) \cos(ky) + \mathrm{h.o.t.} \tag{4.16}$$

(*or with the cos ky replaced by* sin *ky*). *This wave can be considered locally a superposition of the wave pair of* (*a*)).

(iii) *Only one of the wave types named in* (i) *and* (ii) *can be stable* (*depending on the parameters*). *When they change stability, this means that a secondary bifurcation to a quasi-periodic solution takes place.*

Finally, it is seen from the above formulae (*or from* (4.7)) *that* $\tilde{k}$ *is related to the transverse period by*

$$\lambda\tilde{k} = k. \tag{4.17}$$

An oblique wave travelling diagonally to the right is represented in figure 4. The symmetry patterns and the approximate shape of the standing travelling waves are illustrated in figure 5. Of course, they rely on the linear theory; it has to be determined numerically what these 'bubbles' actually look like.

4.4. *Some mathematical problems*

In contrast to the corresponding positive results for incompressible one-phase flows, the existence of standing travelling waves has not yet been found owing to mathematical difficulties. We have already described some of these difficulties above.

To go into more detail here, let us first state that we have been able to prove the existence (at least for small times) of a unique solution to the initial-boundary-value problem corresponding to the reduced model described by (2.14) (Göz 1991). Next, the bifurcation to vertical or oblique travelling waves has been shown without referring to the special ansatz (4.3). For this a transformation to Lagrangian coordinates had to be carried out in order to circumvent the difficulties originating from the hyperbolic part of the equations, which was then reduced to an ordinary

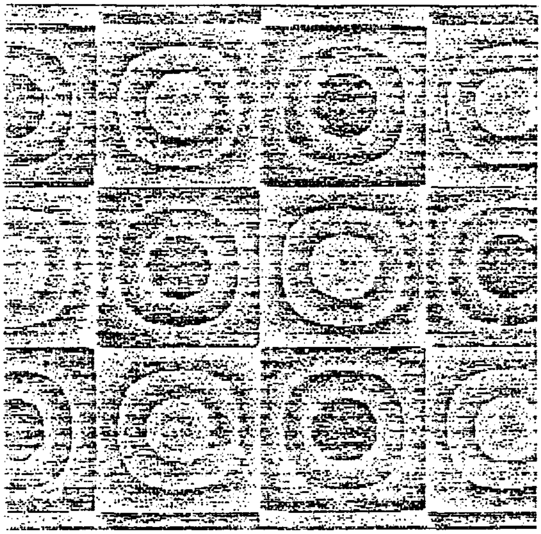

FIGURE 5. Symmetry pattern of a standing travelling wave.

differential equation. In these coordinates, the linearization is a well-defined procedure (thus justifying the analysis of §3), but again the linear non-homogeneous problem cannot be solved in the two-dimensional case. This means, that the Fredholm property does not apply, and the existence of the standing travelling waves cannot be proved. Nevertheless, if we modify the system slightly by subtracting the temporal mean values of all expressions occurring in the equations, it is possible to show the bifurcation of oblique and standing travelling waves for this new system. These solutions correspondingly exist in a slightly different function space, namely with the temporal mean value zero (Göz 1990*b*). If it could be demonstrated that the subtracted expressions yield zero for the solutions of the modified system, then these solutions would actually represent the desired solutions of the original system.

As a consequence, it may well be that the STWs do not exist which would be as strange as it was interesting in view of the usual results in the presence of symmetry. This, however, might indeed happen, as can be seen in the following approach. Consider the parabolic regularization of (2.14*a*), i.e.

$$-\partial_t \phi + \boldsymbol{\nabla} \cdot [(1-\phi)\, \boldsymbol{v}] = -\epsilon \nabla^2 \phi, \tag{4.18}$$

where $\epsilon > 0$ is a small parameter. This yields a parabolic system with a better solution ρ (as regular as $\boldsymbol{v}$). It is known that the desired bifurcation theorem applies for this. Equivalently we can consider the elliptic regularized version of (4.5*a*):

$$\boldsymbol{\nabla} \cdot [(1-\phi)\,(\boldsymbol{v} - \omega \boldsymbol{k})] = -\epsilon \nabla^2 \phi. \tag{4.19}$$

Since the regularized system, when considered a new model, does not make much sense in the present context, we would have to investigate the limit $\epsilon \to 0$. Of course, this is the point where the next difficulty arises.

A short calculation, e.g. for (4.19), shows that a small ϵ induces only small perturbations in the bifurcation curves of §4.2. In particular, the bifurcation points are located in a neighbourhood of order ϵ of those points, which are considered as possible bifurcation points in the case of $\epsilon = 0$. This suggests that the bifurcation scenario might be the same in both cases.

It may very well happen, however, that one or other bifurcation branch vanishes in the limit $\epsilon \rightarrow 0$. Certainly the branches representing the oblique travelling waves do not vanish, since that are plane waves, whose existence has been proved definitely and in two different ways. But we are not able to control the important branch of standing travelling waves. Nevertheless, note what can happen in the limit process, if the bifurcation equations are considered.

With regard to the present symmetries, the bifurcation equations for $\epsilon > 0$ assume the following form to first approximation (cf. e.g. Erneux 1981):

$$\left.\begin{aligned} z(-\mathrm{i}\omega+\alpha\lambda+A|z|^2+B|w|^2) &= 0, \\ \bar{w}(-\mathrm{i}\omega+\alpha\lambda+B|z|^2+A|w|^2) &= 0, \end{aligned}\right\} \tag{4.20}$$

where A and B are two complex non-zero functions of the physical parameters of the system.

If $\mathrm{Re}\,(A+B) \neq 0$ and $\mathrm{Re}\,(A-B) \neq 0$, three solutions of (4.20) exist: the two OTWs determined by $|z| = 0$ or $|w| = 0$, respectively, and the STW given by

$$|z|^2 = |w|^2 = -\mathrm{Re}\,(\alpha)\,\lambda/\mathrm{Re}\,(A+B).$$

However, if $\mathrm{Re}\,(A+B) \rightarrow 0$ as $\epsilon \rightarrow 0$, then only the OTW solutions remain, whereas the STW branch vanishes. In this case the higher approximations would have to be considered which would be a formidable task. If, on the other hand, $\mathrm{Re}\,(A+B)$ does not vanish in the limit $\epsilon \rightarrow 0$, this would strongly support the conjecture that the bifurcation scenario remains the same.

Otherwise this would be an interesting example for which the conclusions from linear theory fail to hold in the nonlinear regime, although such conclusions are valid for nearby systems. In any case, this shows that one has to be very careful in transferring results from linear to nonlinear theory of partial differential equations. We are presently investigating this relationship using the regularizations discussed above.

5. Bifurcations of higher order

So far, we have taken ω as the bifurcation parameter such that unique wavenumbers λ and k were obtained for given ω. It is, however, completely reasonable to interchange the role of ω and k. So let k be a given wavenumber (arising from the given data such as the period in the spanwise direction) and search for those ω (and then for λ) for which (4.9a, b) can be fulfilled, i.e. for which the flow becomes periodic in the streamwise direction, too. On the basis of the results of the preceding section (cf. figure 3), we see that not every ω can be obtained by variations of k, and indeed there are zero to three solutions depending not only on k but also on the constellation of the other parameters of the system. In addition, there are degenerate points with $P'(\omega) = 0$ where the number of solutions changes.

For example let us consider case (IV) of figure 3. Starting with a large k, exactly one wave speed $h < \omega \leqslant d$ is obtained as solution in the above sense. Some wave speeds in the interval (h, d) are excluded if we do not allow the zeros of the function f to wander out of this interval; this would first lead to case (III) and then to (IIb). Another change of c, h, and/or m would eventually lead to a one-to-one correspondence between k^2 and ω as in case (IIa). With decreasing $|k|$ (resp. k^2) (by varying the bed width α, see (4.4)), a point of degeneracy is reached, from which two possibly connected continua of solutions emanate (cf. figure 6a).

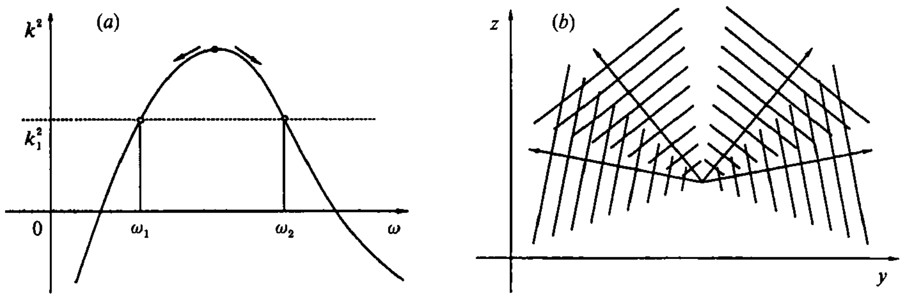

FIGURE 6. Degenerate bifurcation and its unfolding. (a) The appearance of two TW solutions at a point of degeneracy. (b) The interaction of plane waves originated by the process illustrated in (a) propagating in different directions (schematically) (same k gives different ω, thus different λ, hence different $\tilde{k}$).

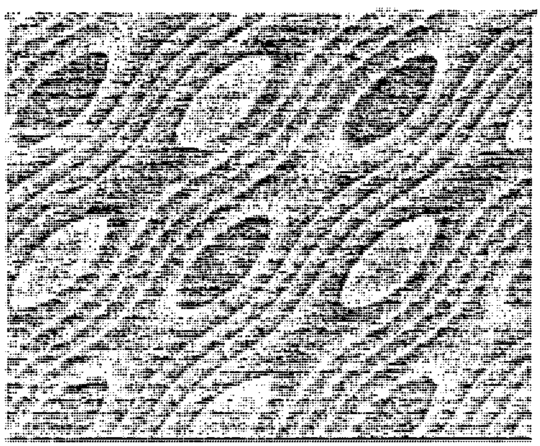

FIGURE 7. Symmetry pattern of a secondary bifurcating solution due to mode interaction.

Considering these results together with other known results on degenerate Hopf bifurcations and their unfoldings (Kielhöfer 1979; Golubitsky & Roberts 1987), possible interactions of these two branches leading to quasi-periodic solutions may be suspected. For the corresponding OTWs this means that their streamwise wavenumber λ is different owing to the difference in ω (cf. figure 2), even though their transverse periodicity is the same. Via the relation (4.17) this results in a more or less different direction of propagation of these plane waves (with different velocities!), which in turn leads to certain interference patterns. This is shown schematically in figure 6(b).

Next, quasi-periodic solutions could also be obtained with three underlying frequencies, i.e. a 3-torus of solutions, by interaction of one of the TWs of one branch with the quasi-periodic solution with two different frequencies (case iii of the last statement of §4) of the other branch. Even higher degeneracies (or their unfoldings in turn) are possible when a parameter constellation is assumed, for which not only the first, but also the second derivative of $P(\omega)$ vanishes. These cases will not be investigated any further here. The symmetry pattern of a secondary bifurcating wave corresponding to a mode interaction between an STW and an OTW travelling diagonally to the right is plotted in figure 7. It must be added that even if the STWs

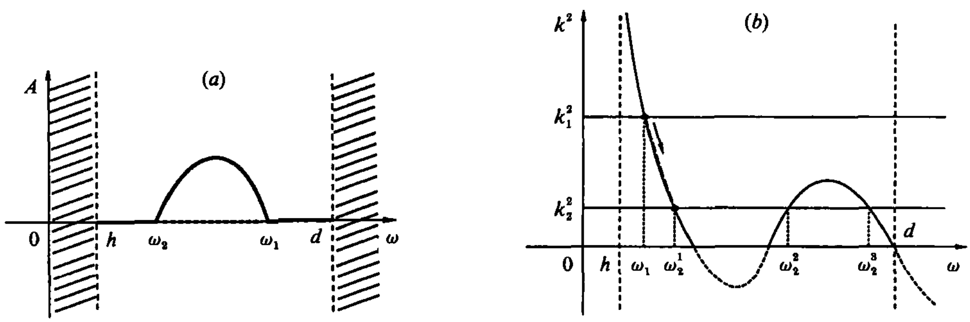

FIGURE 8. Connected continua of periodic solutions. (*a*) Re-entering of bifurcating branches. (*b*) Critical values of ω as a function of k^2: OTW connections for different k (keep $\tilde{k}$ and change ω means that λ changes, k changes; cf. (4.17)).

do not exist, the interaction of the various OTWs could nevertheless lead to secondary etc. bifurcations yielding patterns which are similar to those of the STWs. Thus, more and more complex solution patterns are obtained until a point of transition to turbulence is reached.

Another possibility would be that the two solutions, which are obtained for given k and connected by the degeneracy, correspond to one and the same (nonlinear) solution branch. This means that the periodic solution bifurcates from the basic state at some critical value ω_1 and re-enters at another critical value ω_2 with both values of ω depending on the same k but yielding different λ (see figure 8*a*).

In fact, such a scenario was obtained by Göz (1990*a*, *b*) for the bifurcation of periodic oblique and/or vertical travelling waves. We now want to point out that this is only possible if $\delta \neq 0$! However, in this case a travelling wave system is used where the solutions depend only on the variable $\tilde{z} := x - \omega t \pm \tilde{k}y$, and results might have to be interpreted differently.

In fact, in this case we consider plane waves where $\tilde{k}$, i.e. the direction of propagation, is given (even though $\tilde{k}$ may be considered an additional parameter), and the existence and connection of periodic solutions are studied with respect to the main bifurcation parameter ω. If we keep the direction of propagation and vary the propagation velocity ω, this results in a continuous change of the spatial periods of the solution and in particular corresponds to changes in the transverse periodicity which can be seen by the relation (4.17).

Of course this must match with the geometry of the underlying domain but even then several solutions with multiple periods may be possible and, more generally, when searching for all solutions possible in principle of such a type admitted by the equations, we could regard the geometry of the system itself as a variable parameter (as α above) to gain the desired results.

As a result, we see that the second possibility contains a connection *along* the curve $k^2 \sim P(\omega)$ and not between certain points at the intersection of this curve with a line $k^2 = \text{const.}$ (cf. figure 8*b*). This will be dealt with in more detail elsewhere.

More detailed results on the bifurcating travelling and standing waves and their interrelated stability properties, especially a possible exchange of stabilities leading to quasi-periodic solutions, and the connection of branches emanating from degenerate points, require further numerical or analytical work. We are currently doing some of this.

In particular, we have calculated the linear stability of a vertical travelling plane wave to transverse perturbations in the case of $\delta = \nu = 0$ (announced in Göz 1990*a*, details unpublished). The results indicate a secondary stationary bifurcation to another branch of periodic solutions of the form

$$U(x,y,t) = F(x-\omega t)\,e^{iky} + \text{c.c.} + \text{h.o.t.} \tag{5.1}$$

Obviously, these waves have a structure that is similar to that of the above-mentioned standing travelling waves; but their origin is different. Note that this scenario of primary and secondary bifurcations is in accordance with experimental observations (Didwania & Homsy 1981 *a*, *b*) as well as with recent numerical results (Hernandez & Jimenez 1990).

6. Conclusions

We have investigated a model based on the theory of interacting continua for the description of the flow of two miscible phases in a fluidized bed from a bifurcation point of view. Although no mathematically rigorous solution of the existence and bifurcation problem corresponding to the full equations (2.9) has been found up to now, we though it necessary to analyse the linearized travelling wave problem in order to see what kind of solutions are principally allowed within this model. Indeed we have seen that the equations may possess a rich variety of solutions, which will be investigated in more detail in the near future.

In particular, we obtained the existence of vertical and oblique travelling waves, whose interactions can lead to higher instabilities and, subsequently, to the occurrence of patterns of increasing complexity. This would be even more the case if the so-called standing travelling waves could be shown to exist. A partial result in this direction was obtained for the case of $\delta = \nu = 0$. This point, however, requires further research.

Moreover, since the main instability leads to vertical travelling plane waves, all the other primary bifurcating branches with non-trivial transverse structure may be unstable, but gain stability by secondary or higher-order bifurcations. These scenarios have to be evaluated in more detail with analytical as well as numerical methods. Another possibility consists in a secondary bifurcation of the vertical travelling wave solution to some other standing travelling wave; indeed, this is the scenario usually observed in experiments. Again, further instabilities are likely to occur. In any case, this leads to waves containing several different frequencies and finally to chaotic behaviour.

The computation of the actual shape of all these solutions and the comparison with the experiments is of particular interest. However, since the principal behaviour of the system does not depend strongly on the detailed modelling – for instance of the drag force or of voidage-dependent viscosity coefficients – it may be difficult to choose among the different models.

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REFERENCES

Batchelor, G. K. 1988 A new theory of the instability of a uniform fluidized bed. *J. Fluid Mech.* **193**, 75–110.

BRIDGES, T. J. 1989 The Hopf bifurcation theorem with symmetry for the Navier–Stokes equations in $(L_p(\Omega))^n$, with application to plane Poiseuille flow. *Arch. Rat. Mech. Anal.* **106**, 335–376.

CHOQUET-BRUHAT, Y., DEWITT-MORETTE, C. & DILLARD-BLEICK, M. 1982 *Analysis, Manifolds and Physics*. North-Holland.

CHOSSAT, P. & IOOSS, G. 1985 Primary and secondary bifurcations in the Couette–Taylor problem. *Japan J. Appl. Maths* **2**, 37–68.

DIDWANIA, A. K. & HOMSY, G. M. 1981*a* Flow regimes and flow transitions in liquid fluidized beds. *Intl. J. Multiphase Flow* **7**, 563–580.

DIDWANIA, A. K. & HOMSY, G. M. 1981*b* Rayleigh–Taylor instabilities in fluidized beds. *Indust. Engng Chem. Fundam.* **20**, 318–323.

DIDWANIA, A. K. & HOMSY, G. M. 1982 Resonant side-band instabilities in wave propagation in fluidized beds. *J. Fluid Mech.* **122**, 433–438.

DREW, D. A. 1983 Mathematical modelling of two-phase flow. *Ann. Rev. Fluid Mech.* **15**, 261–91.

ERNEUX, T. 1981 Stability of rotating chemical waves. *J. Math. Biol.* **12**, 199–214.

ERNEUX, T. & MATKOWSKY, B. J. 1984 Quasi-periodic waves along a pulsating propagating front in a reaction-diffusion equation. *SIAM J. Appl. Maths* **44**, 536–544.

GANSER, G. H. & DREW, D. A. 1987 Nonlinear periodic waves in a two-phase flow model. *SIAM J. Appl. Maths* **47**, 726–736.

GANSER, G. H. & DREW, D. A. 1990 Nonlinear stability analysis of a uniformly fluidized bed. *Intl. J. Multiphase Flow* **16**, 447–460.

GARG, S. K. & PRITCHETT, J. W. 1975 Dynamics of gas-fluidized beds. *J. Appl. Phys.* **46**, 4493–4500.

GOLUBITSKY, M. & ROBERTS, M. 1987 A classification of degenerate Hopf bifurcation with $O(2)$ symmetry. *J. Diffl Equat.* **69**, 216–264.

GOLUBITSKY, M. & STEWART, I. 1985 Hopf bifurcation in the presence of symmetry. *Arch. Rat. Mech. Anal.* **87**, 107–165.

GOLUBITSKY, M. & STEWART, I. 1986 Symmetry and stability in Taylor–Couette flow. *SIAM J. Math. Anal.* **17**, 249–288.

GÖZ, M. F. 1990*a* Instabilities and bifurcations in a two-dimensional fluidized bed model. *Z. Angew. Math. Mech.* **70**, T 386–T 388.

GÖZ, M. F. 1990*b* Bifurcation analysis of a two-dimensional fluidized bed model. Ph.D. thesis, Universität Heidelberg.

GÖZ, M. F. 1991 Existence and uniqueness of time-dependent spatially periodic solutions of fluidized bed equations. *Z. Angew. Math. Mech.* **71**, T 754-T 755.

GRUBB, G. & GEYMONAT, G. 1977 The essential spectrum of elliptic systems of mixed order. *Math. Annln* **227**, 247–276.

HERNANDEZ, J. A. & JIMENEZ, J. 1991 Bubble formation in dense fluidised beds. *Phys. Fluids* A **3**, 1457.

HOMSY, G. M. 1983 A survey of some results in the mathematical theory of fluidization. In *Theory of Dispersed Multiphase Flow*, pp. 57–71. Academic.

IOOSS, G. 1984 Bifurcation and transition to turbulence in hydrodynamics. In *Bifurcation Theory and Applications*. Lecture Notes in Mathematics, vol. 1057, pp. 152–201. Springer.

JACKSON, R. 1971 Fluid mechanical theory. In *Fluidization* (ed. J. F. Davidson & D. Harrison), pp. 65–119. Academic.

KIELHÖFER, H. 1979 Generalized Hopf bifurcation in Hilbert space. *Math. Meth. Appl. Sci.* **1**, 498–513.

KLUWICK, A. 1983 Small-amplitude finite-rate waves in suspensions of particles in fluids. *Z. Angew. Math. Mech.* **63**, 161–171.

KURDYUMOV, V. N. & SERGEEV, YU. A. 1987 Propagation of nonlinear waves in a fluidized bed in the presence of interaction between the particles of the dispersed phase. *Fluid Dyn.* **22**, 235–242.

LIU, J. T. C. 1982 Note on a wave-hierarchy interpretation of fluidized bed instabilities. *Proc. R. Soc. Lond.* A **380**, 229–239.

LIU, J. T. C. 1983 Nonlinear unstable wave disturbances beds. *Proc. R. Soc. Lond.* A **389**, 331–347.

MEDLIN, J., WONG, H.-W. & JACKSON, R. 1974 Fluid mechanical description of fluidized beds. Convective instabilities in bounded beds. *Indust. Engng Chem. Fundam.* **13**, 247–259.

NEEDHAM, D. J. & MERKIN, J. H. 1983 The propagation of a voidage disturbance in a uniformly fluidized bed. *J. Fluid Mech.* **131**, 427–454.

NEEDHAM, D. J. & MERKIN, J. H. 1984*a* The evolution of a two-dimensional small-amplitude voidage disturbance in a uniformly fluidized bed. *J. Engng Maths.* **18**, 119–132.

NEEDHAM, D. J. & MERKIN, J. H. 1984*b* A note on the stability and the bifurcation to periodic solutions for wave-hierarchy problems with dissipation. *Acta Mech.* **54**, 74–85.

NEEDHAM, D. J. & MERKIN, J. H. 1986 The existence and stability of quasi-steady periodic voidage waves in a fluidized bed. *Z. Angew. Math. Phys.* **37**, 322–339.

PRITCHETT, J. W., BLAKE, T. R. & GARG, S. K. 1978 A numerical model of gas fluidized beds. *AIChE Symp. Ser.* **74**, 134–148.

RICHARDSON, J. E. 1971 In *Fluidization* (ed. J. F. Davidson & D. Harrison), Academic.

SATTINGER, D. H. 1983 *Branching in the Presence of Symmetry.* SIAM Monograph CMBS-NSF, Series No. 40.

SERGEEV, YU. A. 1990 Steady concentration waves and dispersion effects in a gas-particle two-phase medium with weak particle interaction. *Fluid Dyn.* **25**, 34–39.

SPIEGEL, E. A. & CHILDRESS, W. S. 1975 Archimedean instabilities in two-phase flows. *SIAM Rev.* **17**, 136–165.

VANDERBAUWHEDE, A. 1982 *Local Bifurcation and Symmetry.* Research Notes in Mathematics, vol. 75. Pitman.